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Exact wavefunctions of the time-dependent damped harmonic oscillator with an arbitrary varying mass and with a force quadratic in velocity under the action of an arbitrary time-varying driving force

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Abstract. This article adopts a Gaussian-type propagator to find the exact wavefunction of a very general time-dependent damped harmonic oscillator with an arbitrary varying mass and with a force quadratic in velocity under the action of an arbitrary time-varying driving force. The results obtained not only generalize all the known results in the literatures but can also be applied to many interesting particular cases.

1. Introduction

Recently we have discussed the invariants and symmetries for a particle with a force quadratic in velocity [1], and the Noether's theorem invariants for a time-dependent damped harmonic oscillator with a force quadratic in velocity [2]. We have also discussed the propagator and exact wavefunctions of the harmonic oscillator with strongly pulsating mass under the action of an arbitrary driving force [3]. On the basis of these works, we shall now discuss the propagator and exact wavefunctions of a rather general system for a damped time-dependent harmonic oscillator with an arbitrary varying mass and with a force quadratic in velocity, furthermore, we consider this system is under the action of an arbitrary time-varying driving force. Since this system is very general, we can apply the results obtained to many interesting particular cases.

2. Equation of motion and Hamiltonian

The equation of motion of the above-mentioned system is

$$\ddot{x} + \beta_1 \dot{x} + \tfrac{1}{2}\gamma \dot{x}^2 + \frac{\partial V}{\partial x} = \frac{f(x,t)}{M(t)} \tag{1}$$

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where

$$\beta_1 = \beta + \frac{\dot{M}(t)}{M(t)} \tag{2}$$

the admissible potential V is chosen as

$$\frac{\partial V}{\partial x} = \omega^2(t)\frac{1-\exp\left[-\left(\frac{\gamma}{2}\right)x\right]}{\gamma/2} \tag{3}$$

the time-dependent driving force is chosen as

$$f(x,t) = \frac{M(t)}{M_0}F(t)\exp\left[-\left(\frac{\gamma}{2}\right)x\right] \tag{4}$$

$M(t)$ is the arbitrary time-varying mass, $\omega(t)$ is the arbitrary time-varying angular frequency, $F(t)$ is an arbitrary time-dependent function, $M_0 = M(0)$, β and γ are constants.

The Hamiltonian for such a system is found to be

$$H = \frac{p^2}{2M}e^{-(\beta t+\gamma x)} + \frac{M}{2}\omega^2 e^{\beta t}\left(\frac{e^{(\gamma/2)x}-1}{\gamma/2}\right)^2 - \frac{M}{M_0}F(t)e^{\beta t}\frac{e^{(\gamma/2)x}-1}{\gamma/2}. \tag{5}$$

When we perform the canonical transformation:

$$q = \frac{e^{(\gamma/2)x}-1}{\gamma/2} \qquad p_1 = e^{-(\gamma/2)x}p \tag{6}$$

equation (5) becomes

$$H_1 = \frac{p_1^2}{2M}e^{-\beta t} + \frac{M}{2}\omega^2 e^{\beta t}q^2 - \frac{M}{M_0}F(t)e^{\beta t}q. \tag{7}$$

Performing further the following canonical transformation:

$$Q = \left(\frac{M}{M_0}\right)^{1/2} q \qquad P = \left(\frac{M_0}{M}\right)^{1/2} p_1 \tag{8}$$

we obtain the new Hamiltonian:

$$H_2 = \frac{P^2}{2M_0}e^{-\beta t} + \frac{M_0}{2}\omega^2 e^{\beta t}Q^2 + \frac{\dot{M}}{4M}(QP+PQ) - \left(\frac{M}{M_0}\right)^{1/2} F(t)e^{\beta t}Q. \tag{9}$$

In the particular case $\beta = \gamma = 0$, $M(t) = M_0\cos^2(\nu t)$, equation (9) reduces to equation (4) of [3]. Details of the derivation of (9) are shown in appendix 1.

The corresponding Hamiltonian operator of (9) is

$$\hat{H}_2 = -\frac{\hbar^2}{2M_0}\frac{\partial^2}{\partial Q^2}e^{-\beta t} + \frac{M_0}{2}\omega^2 e^{\beta t}Q^2 + \frac{\dot{M}}{4M}\left(-i\hbar Q\frac{\partial}{\partial Q} - i\hbar\frac{\partial}{\partial Q}Q\right) - \left(\frac{M}{M_0}\right)^{1/2} F(t)e^{\beta t}Q. \tag{10}$$

3. Propagator

We adopt a Gaussian-type propagator K to solve the time-dependent Schrödinger equation,

$$K(Q,t;Q_0,0) = A_0 \exp[-C_1Q^2 - C_2Q - (C_4Q + C_5)Q_0 - C_3Q_0^2] \quad (11)$$

where

$$Q_0 = Q(0) = \frac{e^{(\gamma/2)x_0} - 1}{\gamma/2} \qquad x_0 = x(0). \quad (12)$$

The propagator satisfies the wave equation

$$i\hbar\frac{\partial}{\partial t}K = \hat{H}_2K. \quad (13)$$

Substituting (11) into (13) and comparing the coefficients of the different powers of Q and Q_0, we obtain

$$-i\hbar\dot{C}_1 = -aC_1^2 + \frac{M_0}{2}\omega^2 e^{\beta t} + ibC_1 \qquad a = \frac{2\hbar^2 e^{-\beta t}}{M_0} \qquad b = \frac{\hbar\dot{M}}{M} \quad (14)$$

$$-i\hbar\dot{C}_2 = -aC_1C_2 + \frac{ib}{2}C_2 - \left(\frac{M}{M_0}\right)^{1/2} F(t)e^{\beta t} \quad (15)$$

$$-i\hbar\dot{C}_3 = -\frac{a}{4}C_4^2 \quad (16)$$

$$-i\hbar\dot{C}_4 = -aC_1C_4 + \frac{ib}{2}C_4 \quad (17)$$

$$-i\hbar\dot{C}_5 = -\frac{a}{2}C_2C_4 \quad (18)$$

$$i\hbar\frac{dA_0}{dt} = \left(\frac{a}{2}C_1 - \frac{a}{4}C_2^2 - \frac{ib}{4}\right)A_0. \quad (19)$$

Integrating (14), we obtain (see appendix 2)

$$C_1 = \frac{M_0 e^{\beta t}}{2i\hbar}\left[\frac{\Omega_0\cot(\Omega_0 s)}{\rho^2} + \frac{\dot{\rho}}{\rho} - \frac{\beta}{2} - \frac{\dot{M}}{2M}\right] \quad (20)$$

where

$$\Omega^2 = \left(\omega^2 - \frac{\beta\dot{M}}{2M} - \frac{\ddot{M}}{2M} + \frac{\dot{M}}{4M^2}\right) - \frac{1}{4}\beta^2 \qquad \ddot{\rho} + \Omega^2\rho = \frac{\Omega_0^2}{\rho^3} \quad (21)$$
$$\dot{s} = \bar{\rho}^2 \qquad \Omega_0 = \Omega(0).$$

Substituting (20) into (15) we obtain

$$\dot{C}_2 = -\left(\frac{\Omega_0\cot(\Omega_0 s)}{\rho^2} + \frac{\dot{\rho}}{\rho} - \frac{\beta}{2}\right)C_2 - \left(\frac{M}{M_0}\right)^{1/2} F(t)e^{\beta t}. \quad (22)'$$

Multiplying the integrating factor $\rho\sin(\Omega_0 s)\exp(-(\beta/2)t)$ at both sides of the above equation and integrating we get

$$C_2 = \frac{M_0 e^{\beta t/2}}{i\hbar\rho\sin(\Omega_0 S)}\int_0^t e^{\beta t/2}\left(\frac{M}{M_0}\right)^{1/2} F(\tau)\rho\sin[\Omega_0 s(\tau)]\,d\tau. \quad (22)$$

Substituting (20) into (17) we obtain

$$\frac{\dot{C}_4}{C_4} = \frac{\beta}{2} - \frac{\dot{\rho}}{\rho} - \frac{\Omega_0 \cot(\Omega_0 s)}{\rho^2}. \tag{23$'$}$$

Integrating the above equation we get

$$C_4 = \frac{-\Omega_0 M_0 \exp(\beta t/2)}{i\hbar \rho \sin(\Omega_0 s)}. \tag{23}$$

Substituting (23) into (16) and integrating it, we obtain

$$C_3 = \frac{\Omega_0 M_0}{2i\hbar} \xi(s) \tag{24}$$

where

$$\xi(s) = \frac{\beta}{2\Omega_0} + \cot(\Omega_0 s). \tag{25}$$

Substituting (22), (23) into (18) and integrating it we get (see appendix 3)

$$C_5 = \frac{M_0}{i\hbar \sin(\Omega_0 s)} \int_0^t e^{\frac{\beta\tau}{2}} \left(\frac{M}{M_0}\right)^{1/2} F(\tau)\rho \sin[\Omega_0(s(t) - s(\tau))]\, d\tau. \tag{26}$$

Substituting (20), (22) into (19) we obtain

$$\frac{dA_0}{A_0} = \left(-\frac{\Omega_0 \cot(\Omega_0 s)}{2\rho^2} - \frac{\dot{\rho}}{2\rho} + \frac{\beta}{4}\right) dt + \frac{M_0\, dt}{2i\hbar \rho^2 \sin^2(\Omega_0 s)} \left(\int_0^t e^{(\beta/2)\tau} \left(\frac{M}{M_0}\right)^{1/2} F(\tau)\rho \sin[\Omega_0 s(\tau)]\, d\tau\right)^2. \tag{27$'$}$$

Integrating the above equation we get

$$A_0 = \left(\frac{M_0 \Omega_0 e^{\beta t/2}}{2\pi i\hbar \rho \sin(\Omega_0 s)}\right)^{1/2} \exp\left[\frac{M_0}{2i\hbar} \int_0^t \frac{1}{\rho^2 \sin^2(\Omega_0 s)} \times \left(\int_0^t e^{\beta\tau/2} \left(\frac{M}{M_0}\right)^{1/2} F(\tau)\rho \sin(\Omega_0 s(\tau))\, d\tau\right)^2 dt\right]. \tag{27}$$

In the particular case $\beta = \gamma = 0$, $M(t) = M_0 \cos^2(\nu t)$, equations (20), (22), (23), (24), (26) and (27) reduce to equation (9) of [3].

For convenience we put

$$R(t) = \frac{e^{-\beta t/2}}{\Omega_0} \int_0^t e^{\beta\tau/2} \left(\frac{M}{M_0}\right)^{1/2} F(\tau)\rho \sin[\Omega_0(s(t) - s(\tau)]\, d\tau. \tag{28}$$

In the particular case of [3], equation (28) reduces to equation (10) of [3].

Substituting (28) into (26) and (22), we obtain

$$C_5 = \frac{M_0 \Omega_0 e^{\beta t/2}}{i\hbar \sin(\Omega_0 s)} R(t) \tag{29}$$

$$C_2 = \frac{M_0 \rho e^{\beta t}}{i\hbar} \left\{\left[\frac{\beta}{2} - \frac{\Omega_0 \cot(\Omega_0 s)}{\rho^2}\right] R(t) + \dot{R}(t)\right\}. \tag{30}$$

Using (30) we can rewrite (27) as

$$A_0 = \left(\frac{M_0\Omega_0 e^{\beta t/2}}{2\pi i\hbar\rho\sin(\Omega_0 s)}\right)^{1/2} \exp\left\{\frac{M_0}{2i\hbar}\int_0^t \rho^2 e^{\beta t}\left[\left(\frac{\beta}{2}-\frac{\Omega_0\cot(\Omega_0 s)}{\rho^2}\right)R+\dot{R}\right]^2 dt. \tag{31}$$

Substituting (20), (23), (24), (29), (30) and (31) into (11) we get

$$\begin{aligned} K(Q,t;Q_0,0) = {} & \left(\frac{M_0\Omega_0\exp(\beta t/2)}{2\pi i\hbar\rho\sin(\Omega_0 s)}\right)^{1/2} \\ & \times\exp\left[\left(\frac{M_0\Omega_0}{2i\hbar}\right)\left\{\frac{\exp(\beta t)}{\Omega_0}\left[\frac{\beta}{2}-\frac{\dot{\rho}}{\rho}-\frac{\Omega_0\cot(\Omega_0 s)}{\rho^2}+\frac{\dot{M}}{2M}\right]Q^2\right.\right. \\ & -\frac{2\rho\exp(\beta t)}{\Omega_0}\left[\left(\frac{\beta}{2}-\frac{\Omega_0\cot(\Omega_0 s)}{\rho^2}\right)R+\dot{R}\right]Q \\ & +\frac{2\exp(\beta t/2)}{\rho\sin(\Omega_0 s)}(Q-\rho R)Q_0-\xi(s)Q_0^2 \\ & \left.\left.+\int_0^t\frac{\rho^2\exp(\beta t)}{\Omega_0}\left[\left(\frac{\beta}{2}-\frac{\Omega_0\cot(\Omega_0 s)}{\rho^2}\right)R+\dot{R}\right]^2 dt\right\}\right]. \end{aligned} \tag{32}$$

In the particular case of [3], equation (32) reduces to equation (15) of [3]. We have also verified that, in the particular case of [4], equation (32) reduces to equation (2.5) of [4]. By the way, since (32) is derived from (13), at $t = 0$ the propagator $K(Q,0;Q_0,0) = \delta(Q-Q_0)$ [5].

4. Wavefunctions

The wavefunctions are calculated using the formula

$$\psi_n(Q,t) = \int_{-\infty}^{\infty} dQ_0\, K(Q,t;Q_0,0)\psi_n(Q_0,0) \tag{33}$$

where $\psi_n(Q_0,0)$ is the wavefunction for a simple harmonic oscillator at $t = 0$ (see appendix 4)

$$\psi_n(Q_0,0) = \left(\frac{\sqrt{M_0\Omega_0/\hbar}}{2^n n!\sqrt{\pi}}\right)^{1/2} H_n\left(\sqrt{\frac{M_0\Omega_0}{\hbar}}Q_0\right)\exp\left(\frac{-M_0\Omega_0}{2\hbar}Q_0^2\right) \tag{34}$$

and H_n is the usual Hermite polynomial. Substituting (32) and (34) into (33) we obtain

$$\psi_n(Q,t) = \left(\frac{D}{2^n n!\sqrt{\pi}}\right)^{1/2} I_n\exp[B_1Q^2+B_2Q+B_3] \tag{35}$$

where

$$\begin{aligned} I_n = {} & \left(\frac{M_0\Omega_0\sqrt{1+\xi^2}}{2\pi i\hbar}\right)^{1/2}\int_{-\infty}^{\infty}\exp\left\{-\frac{M_0\Omega_0}{2\hbar}(1-i\xi)\left[Q_0-\frac{e^{\beta t/2}((Q/\rho)-R)}{(1-i\xi)i\sin(\Omega_0 s)}\right]^2\right\} \\ & \times H_n\left(\sqrt{\frac{M_0\Omega_0}{\hbar}}Q_0\right)dQ_0 \end{aligned} \tag{36}$$

$$\begin{aligned} & D = \left(\frac{M_0\Omega_0 e^{\beta t}}{\hbar\zeta_1}\right)^{1/2} \qquad B_1 = -\tfrac{1}{2}D^2(1+i\zeta_2) \\ & B_2 = D^2R\rho(1+i\zeta_3) \qquad B_3 = -\tfrac{1}{2}D^2R^2\rho^2(1\mp i\zeta_4) \end{aligned} \tag{37}$$

and

$$\zeta_1 = \rho^2 \sin^2(\Omega_0 s)(1+\xi^2)$$
$$\zeta_2 = \xi + \frac{\zeta_1}{\Omega_0}\left(\frac{\beta}{2} - \frac{\dot\rho}{\rho} - \frac{\Omega_0 \cot(\Omega_0 s)}{\rho^2} + \frac{\dot M}{2M}\right)$$
$$\zeta_3 = \xi + \frac{\zeta_1}{\Omega_0}\left(\frac{\beta}{2} - \frac{\Omega_0 \cot(\Omega_0 s)}{\rho^2} + \frac{\dot R}{R}\right)$$
$$\zeta_4 = \xi + \frac{\zeta_1 e^{-\beta t}}{\Omega_0^2 R^2 \rho^2}\int_0^t \rho^2 e^{\beta t}\left[\left(\frac{\beta}{2} - \frac{\Omega_0 \cot(\Omega_0 s)}{\rho^2}\right) R + \dot R\right]^2 dt. \tag{38}$$

Letting $y = \sqrt{M_0 \Omega_0/\hbar}\, Q_0$ and using the formulae

$$e^{2\tau y - \tau^2} = \sum_{n=0}^{\infty} H_n(y)\frac{\tau^n}{n!}$$
$$\int_{-\infty}^{\infty} \exp[-a(x-b)^2]\, dx = \sqrt{\frac{\pi}{a}}$$
$$\frac{1}{i}\sqrt{\frac{1+i\xi}{1-i\xi}} \equiv \exp\{-i\cot^{-1}[\xi(s)]\} \tag{39}$$

we obtain

$$\sum_{n=0}^{\infty} I_n \frac{\tau^n}{n!} = \frac{1}{\sqrt{i}}\left(\frac{1+i\xi}{1-i\xi}\right)^{1/4} \exp\left\{2D(Q-\rho R)\frac{\tau}{i}\sqrt{\frac{1+i\xi}{1-i\xi}} - \left(\frac{\tau}{i}\sqrt{\frac{1+i\xi}{1-i\xi}}\right)^2\right\}$$
$$= \sum_{n=0}^{\infty} \exp\{-i(n+\tfrac{1}{2})\cot^{-1}\xi\} H_n\{D(Q-\rho R)\}\frac{\tau^n}{n!} \tag{40}$$

hence we get

$$I_n = \exp\{-i(n+\tfrac{1}{2})\cot^{-1}\xi\} H_n\{D(Q-\rho R)\}. \tag{41}$$

Substituting (31) and (41) into (35) and making some simplifications, we get

$$\psi_n(x,t) = \left(\frac{D}{2^n n!\sqrt{\pi}}\right)^{1/2} H_n\left\{D\left[\sqrt{\frac{M}{M_0}}\frac{e^{(\gamma/2)x}-1}{\gamma/2} - \rho R\right]\right\}$$
$$\times \exp\left\{B_1 \frac{M}{M_0}\left(\frac{e^{(\gamma/2)x}-1}{\gamma/2}\right)^2 + B_2\sqrt{\frac{M}{M_0}}\left(\frac{e^{(\gamma/2)x}-1}{\gamma/2}\right)\right.$$
$$\left. + B_3 - i(n+\tfrac{1}{2})\cot^{-1}[\xi(s)]\right\}. \tag{42}$$

Equation (42) generalizes the results which we obtained in [3]; it can be used to study many interesting particular cases. The result given in [4] is the simplest particular case.

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Appendix 1. Derivation of (9)

From (7) we obtain

$$\dot{q} = \frac{\partial H_1}{\partial p_1} = \frac{p_1}{M}\mathrm{e}^{-\beta t} \tag{A1.1}$$

$$\dot{p}_1 = -\frac{\partial H_1}{\partial q} = -M\omega^2\mathrm{e}^{\beta t}q + \frac{M}{M_0}F(t)\mathrm{e}^{\beta t}. \tag{A1.2}$$

From (8) and (6), using (A1.1) and (A1.2) we get

$$\begin{aligned}\dot{Q} &= \left(\frac{M}{M_0}\right)^{1/2}\dot{q} + \frac{1}{2}\left(\frac{1}{M_0 M}\right)^{1/2}\dot{M}q \\ &= \left(\frac{M}{M_0}\right)^{1/2}\mathrm{e}^{-\beta t}\frac{1}{M}\left(\frac{M}{M_0}\right)^{1/2}P + \frac{1}{2}\left(\frac{1}{M_0 M}\right)^{1/2}\dot{M}\left(\frac{M_0}{M}\right)^{1/2}Q \\ &= \frac{P}{M_0}\mathrm{e}^{-\beta t} + \frac{\dot{M}}{2M}Q \\ &= \frac{\partial H_2}{\partial P}\end{aligned} \tag{A1.3}$$

$$\begin{aligned}\dot{P} &= \left(\frac{M_0}{M}\right)^{1/2}\dot{p}_1 - \frac{1}{2}\left(\frac{M_0}{M}\right)^{1/2}\frac{\dot{M}}{M}p_1 \\ &= \left(\frac{M_0}{M}\right)^{1/2}\left[-M\omega^2\mathrm{e}^{\beta t}\left(\frac{M_0}{M}\right)^{1/2}Q + \frac{M}{M_0}F(t)\mathrm{e}^{\beta t}\right] \\ &\quad - \frac{1}{2}\left(\frac{M_0}{M}\right)^{1/2}\frac{\dot{M}}{M}\left(\frac{M}{M_0}\right)^{1/2}P \\ &= -M_0\omega^2\mathrm{e}^{\beta t}Q + \left(\frac{M}{M_0}\right)^{1/2}F(t)\mathrm{e}^{\beta t} - \frac{\dot{M}}{2M}P \\ &= -\frac{\partial H_2}{\partial Q}.\end{aligned} \tag{A1.4}$$

Integrating (A1.3) and (A1.4) we obtain

$$H_2 = \frac{P^2}{2M_0}\mathrm{e}^{-\beta t} + \frac{\dot{M}}{2M}QP + g(Q) \tag{A1.5}$$

$$H_2 = \frac{1}{2}M_0\omega^2\mathrm{e}^{\beta t}Q^2 + \frac{\dot{M}}{2M}PQ - \left(\frac{M}{M_0}\right)^{1/2}F(t)\mathrm{e}^{\beta t}Q + h(P). \tag{A1.6}$$

Comparing (A1.5) and (A1.6) we get

$$g(Q) = \frac{M_0}{2}\omega^2 e^{\beta t} Q^2 - \left(\frac{M}{M_0}\right)^{1/2} F(t) e^{\beta t} Q \qquad h(P) = \frac{P^2}{2M_0} e^{-\beta t}. \quad \text{(A1.7)}$$

Considering the symmetrization rule [6], we write PQ or QP as $\frac{1}{2}(QP + PQ)$. Hence, from (A1.5) or (A1.6) we obtain expression (9). Moreover, the third term on the right-hand side of (9) is in accordance with the corresponding expression in [7, 8].

Appendix 2. Derivation of (20)

Let

$$C_1 = C_1^* - \frac{b}{2ia}. \quad \text{(A2.1)}$$

Substituting (A2.1) into (14) we obtain

$$-i\hbar \dot{C}_1^* = -aC_1^{*2} + \frac{\hbar^2}{a}\left(\Omega^2 + \frac{\beta^2}{4}\right) \quad \text{(A2.2)}$$

where

$$\Omega^2 = \omega^2 - \frac{\beta \dot{M}}{2M} - \frac{\ddot{M}}{2M} + \frac{\dot{M}}{4M^2} - \frac{1}{4}\beta^2. \quad \text{(A2.3)}$$

Let

$$C_1^* = \frac{\hbar}{ia}\left(\frac{\Omega_0 y}{\rho^2} + \frac{\dot{\rho}}{\rho} - \frac{\beta}{2}\right) \qquad (\Omega_0 = \Omega(0)) \quad \text{(A2.4)}$$

where ρ satisfy

$$\ddot{\rho} + \Omega^2 \rho = \frac{\Omega_0^2}{\rho^3}. \quad \text{(A2.5)}$$

Substituting (A2.4), (A2.5) into (A2.2) we get

$$-\dot{y} = \frac{\Omega_0}{\rho^2}(y^2 + 1). \quad \text{(A2.6)}$$

Introduce s and let

$$\dot{s} = \rho^{-2}. \quad \text{(A2.7)}$$

Substituting (A2.7) into (A2.6) and integrating, we obtain

$$y = \cot(\Omega_0 s). \quad \text{(A2.8)}$$

Substituting (A2.4), (A2.8) into (A2.1) we get (20). Equation (21) is given by (A2.3), (A2.5) and (A2.7).

Appendix 3. Derivation of (26)

Substituting (22), (23) into (18), we obtain

$$\dot{C}_5 = \frac{\Omega_0 M_0}{i\hbar\rho^2 \sin^2(\Omega_0 s)} \int_0^t e^{\beta\tau/2} \left(\frac{M}{M_0}\right)^{1/2} F(\tau)\rho \sin[\Omega_0 s(t) - \Omega_0(s(t) - s(\tau))]\,d\tau$$

$$= \frac{\Omega_0 M_0}{i\hbar\rho^2 \sin^2(\Omega_0 s)} \int_0^t e^{\beta\tau/2} \left(\frac{M}{M_0}\right)^{1/2} F(\tau)\rho\{\sin(\Omega_0 s(t)) \cos[\Omega_0(s(t) - s(\tau))] - \cos(\Omega_0 s(t)) \sin[\Omega_0(s(t) - s(\tau))]\}\,d\tau. \tag{A3.1}$$

Letting

$$I = \int_0^t e^{\beta\tau/2} \left(\frac{M}{M_0}\right)^{1/2} F(\tau)\rho \sin[\Omega_0(s(t) - s(\tau))]\,d\tau \tag{A3.2}$$

and using (A2.7) we can rewrite (A3.1) as

$$\dot{C}_5 = \frac{M_0}{i\hbar}\left[I \frac{d}{dt}\left(\frac{1}{\sin(\Omega_0 s)}\right) + \left(\frac{1}{\sin(\Omega_0 s)}\right)\frac{dI}{dt}\right]. \tag{A3.3}$$

Integrating (A3.3) we get

$$C_5 = \frac{M_0 I}{i\hbar \sin(\Omega_0 s)}. \tag{A3.4}$$

Substituting (A3.2) into (A3.4) we readily obtain (26).

Appendix 4. Behaviour of $\psi_n(Q_0, 0)$

Substituting (A1.3) into

$$L_2 = \dot{Q}P - H_2 \tag{A4.1}$$

and eliminating P, we obtain

$$L_2 = \left(\dot{Q} - \frac{\dot{M}Q}{2M}\right)^2 \frac{M_0}{2} e^{\beta t} - \frac{M_0}{2}\omega^2 e^{\beta t} Q^2 + \left(\frac{M}{M_0}\right)^{1/2} F(t) e^{\beta t} Q. \tag{A4.2}$$

Substituting (A4.2) into

$$\frac{d}{dt}\left(\frac{\partial L_2}{\partial \dot{Q}}\right) = \frac{\partial L_2}{\partial Q} \tag{A4.3}$$

and using (21) we get

$$\frac{d^2}{dt^2}(e^{\beta t/2} Q) + \Omega^2 e^{\beta t/2} Q = \left(\frac{M}{M_0}\right)^{1/2} \frac{F(t)}{M_0} e^{\beta t/2}. \tag{A4.4}$$

Since $F(t) = 0$ when $t = 0$, $Q(0) = Q_0$ satisfy

$$\ddot{Q}_0 + \Omega_0^2 Q_0 = 0 \tag{A4.5}$$

i.e. $\psi_n(Q_0, 0)$ is the wavefunction for a simple harmonic oscillator.

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